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GROUP ACTIONS AND A VANISHING CHARACTERISTIC CLASS

LOWELL JONES

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INTRODUCTION

THE PURPOSE of this paper is to show that a certain characteristic class, defined for all Z/p homology manifolds, vanishes if the homology manifold is the fixed point set of a PL Z/p action on a manifold.

Let $\psi : Z/p \times M \to M$ denote a PL action on an oriented PL manifold, and suppose p is an odd prime. Let K denote the fixed point set of ψ, and let $\partial K = K \cap \partial M$. Part of the structure of K is well known:

PROPOSITION. *$(K, \partial K)$ is an orientable Z/p homology manifold pair.*

That K is a Z/p homology manifold was shown by Smith[12]. That it is orientable follows from results of Bredon[2] and Chang and Skjelbred[3].

The characteristic class $h^p = \Sigma_i h_i^p \epsilon \Sigma_i H_{k-1+4i}(K, \partial K; Z)$ is defined (in § 1) by a transversality construction originated by Thom and developed by Morgan and Sullivan[11]. The class corresponds to homomorphisms from PL bordism mod r to the Witt group $W(Z/p)$, defined by an equivariant index.

MAIN THEOREM. *If $(K, \partial K)$ is the fixed point set of a PL Z/p action on an oriented manifold, then $h^p(K) = 0$.*

This result was announced in ([5], Theorem 1.1). Related results have been outlined in[7]. More general characteristic classes have been studied by the author [6, 7] and by Latour[9]. The relationship between h^p and these classes is explained in §3.

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§1. DEFINITION OF THE CHARACTERISTIC CLASS

In outline, the construction is the same as that of Morgan and Sullivan[11].

Recall ([11], § 1) that a PL Z/r manifold $(N; \delta N, \partial N)$ is obtained from a manifold $(\bar{N}, \partial\bar{N})$ by a certain identifications on the boundary. More precisely $\partial\bar{N}$ contains r disjoint copies of δN, and N is obtained by identifying all of these to δN. The remainder of $\partial\bar{N}$ is ∂N, and is also a Z/r manifold. In a similar way Z/r objects are defined in the category of PL Z/p-homology manifolds.

Suppose $(L, \delta L)$ is a closed (i.e. $\partial L = \emptyset$) Z/r object in Z/p-homology manifolds, and suppose there is an orientation $[L] \epsilon H_l(L, \delta L; Z/p)$. An invariant

$$w_p(L) \epsilon Z/2 \tag{1.1}$$

is defined as follows. If $\dim(L) \neq 1 \bmod 4$ then $w_p(L) = 0$. If $\dim(L) = 4q + 1$ then the intersection pairing on the singular set

$$\lambda : H_{2q}(\delta L; Z/p) \times H_{2q}(\delta L : Z/p) \to Z/p$$

(taken with respect to $\partial[L]\epsilon H_{4q}(\delta L; Z/p)$) is a nonsingular symmetric form over the field Z/p. It therefore defines an element in the Witt group (see [4]) $W(Z/p)$.

Recall [4] that there is an exact sequence

$$0 \longrightarrow Z/2 \longrightarrow W(Z/p) \xrightarrow{\substack{\text{rank}\\ \text{mod } 2}} Z/2 \longrightarrow 0.$$

We claim that the rank of $H_{2q}(\delta L; Z/p)$ is even, so $w_p(L)$ can be defined to be $[\lambda]\epsilon$ ker(rank) $\simeq Z/2$. Since p is odd there is an integral class $(L)\epsilon H_{4q+1}(L, \delta L; Z)$ which reduces to a Z/p orientation. $\partial(L)$ defines an intersection pairing

$$H_{2q}(\delta L, Z) \times H_{2q}(\delta L, Z) \to Z$$

which is nonsingular $\otimes Q$. Since $\partial \bar{L} = r\delta L$ ($\bar{L}$ is L split open along δL), this pairing must have 0 signature, and therefore even rank.

This completes the definition of $w_p(L)$.

Next recall [11] that $\Omega_i^{PL}(X, Y; Z/r)$ denotes the group of bordism classes of maps from oriented i-dimensional PL Z/r manifolds to (X, Y).

Suppose $(K, \partial K)$ is a compact PL Z/p-homology manifold pair of dimension k and orientation $[K]$. Embed K as a PL subset of a ball B^s with $\partial K = K \cap \partial B^s$. Choose a regular neighborhood R, and denote by $\partial_0 R$ the topological boundary of R in B^s. Then a homomorphism

$$W_p : \Omega_*^{PL}(R, \partial_0 R; Z/r) \to Z/2 \tag{1.2}$$

is defined as follows. A bordism class is represented by a Z/r manifold $(N; \delta N, \partial N)$ and a map $g:(N, \partial N) \to (R, \partial_0 R)$. Put g in transverse position to all simplices of K. Then $(g^{-1}(K), g^{-1}(K) \cap \delta N)$ is a closed orientable Z/r object in the category of Z/p-homology manifolds. Define $W_p([N, g]) = w_p(g^{-1}(K))$. Notice that this is nonzero only when $\dim N = s - k + 1 \bmod 4$.

LEMMA 1.3. *W_p is a well defined Ω_*^{PL} module homomorphism, when $Z/2$ is given the module structure induced by the index mod 2, $\Omega_*^{PL} \to Z/2$.*

Proof of 1.3. It is helpful to note that $w_p(L)$ is independent of the orientation of L: A second orientation differs from the first by a unit $u\epsilon Z/p$. The intersection matrix differs by u also, so the determinant differs by u^k where k is the rank of $H_{2q}(\delta L; Z/p)$. Since it was shown above that k is even, the two forms define the same element of $W(Z/p)$.

W_p is well defined on bordism classes by the usual argument: Different representatives are bordant. Making the bordism transverse to K gives a Z/p-homology manifold bordism between inverse images of K. The intersection forms are therefore equal in $W(Z/p)$.

To prove that W_p is a module homomorphism, let (N, g) represent a class in $\Omega_n^{PL}(R, \partial_0 R; Z/r)$ and let M be a closed PL manifold. For convenience denote $g^{-1}(K)$ by L. W_p is a homomorphism if $w_p(M \times L) = \text{index}\,(M)\cdot w_p(L)$. The nontrivial case is $\dim(M) = 0 \bmod 4$; the other cases are left to the reader.

Let $\dim M = 4m$. Assume that the intersection form on $H_{2m}(M; Z)/T$ (T = torsion) is indefinite. (It can be made so by surgery on a trivial $2m-1$ sphere.) Then by ([10], Theorems 1 and 2) there is a Z basis with respect to which the intersection matrix $[a_{ij}]$

is diagonal. It follows that $w_p(M \times L) = \Sigma_i a_{ii} w_p(L) \bmod 2$. Since $\Sigma_i a_{ii} = \text{index } (M)$, this proves the claim.

This completes the proof of 1.3.

By ([11], Theorem 7.3) a cohomology class $\sigma(K) \epsilon H^{4^*+s-k+1}(R, \partial_0 R; Z_{(2)})$, where $Z_{(2)}$ is the integers localized at 2, corresponds uniquely to a suitably compatible set of Ω_*^{PL} module homomorphisms

$$\sigma : \Omega^{PL}_{4^*+s-k+1}(R, \partial_0 R : Z) \to Z$$

$$\sigma_k : \Omega^{PL}_{4^*+s-k+1}(R, \partial_0 R : Z/2^k) \to Z/2^k.$$

Define $\sigma = 0$, and σ_k to be the composition of w_p with the inclusion $Z/2 \to Z/2^k$. Then it follows easily from the definition and from 1.3 that these are "suitably compatible".

Definition 1.4. The characteristic class $h^p(K)$ corresponds to the class $\sigma(K)$ under the isomorphisms

$$H^{4^*+s-k+1}(R, \partial_0 R; Z_{(2)}) \simeq H_{4^*+k-1}(R, \partial_1 R; Z_{(2)}) \simeq H_{4^*+k-1}(K, \partial K; Z_{(2)}).$$

Here h^p is defined in homology with $Z_{(2)}$ coefficients. However it has order 2, so is the image of a unique class of order 2 in $H_*(K, \partial K; Z)$. This is the class referred to in the introduction.

§2. PROOF OF THE MAIN THEOREM

Let $K \subset M \subset B^s$ be an embedding with $M \cap \partial B^s = \partial M$, and let R be a regular neighborhood of K in B^s which intersects M in a ψ-invariant regular neighborhood of K. To show that $h^p(K) = 0$ it is sufficient to show that if $g:(N, \partial N) \to (R, \partial_0 R)$ represents a PL Z/r bordism class, then $w_p(g^{-1}(K)) = 0$.

Put g into transverse position to the simplices of both K and $R \cap M$. Then $g^{-1}(R \cap M)$ is an orientable PL Z/r manifold. By ([8], Theorem 1.7) there is a regular neighborhood W of $g^{-1}(K)$ in $g^{-1}(R \cap M)$ and a PL action $\theta : Z/p \times W \to W$ having $g^{-1}(K)$ for fixed set. The theorem therefore follows from the next lemma.

LEMMA 2.1. *Suppose L is the fixed set of a PL Z/p action on a compact oriented Z/r manifold N, and $L \cap \partial N = \emptyset$. Then $w_p(L) = 0$.*

Proof of 2.1. Assume $\dim L = 1 \bmod 4$, since otherwise $w_p(L) = 0$. Assume also that $\dim N = 1 \bmod 4$. In general $\dim N = 1$ or $3 \bmod 4$, but the 3 case can be converted to 1 by crossing with a linear Z/p action on D^2 with fixed set $\{0\}$.

First consider the special case $\partial N = \phi$. Because r copies of δN and $\delta N/\psi$ bound, the indexes of each is zero. Therefore by ([1], Corollary 2), $A(w_p(L)) = 0$, where $A : W(Z/p) \to Z/8$ takes the rank 1 form βX^2 to $3 - 2\,(\beta/p) - p \bmod 8$. Here (β/p) is the Legendre symbol, and is ± 1. It is easily checked that if $A(\alpha) = 0$ and rank $(\alpha) = 0 \bmod 2$, then $\alpha = 0$. The rank statement was proved in (1.1), so $w_p(L) = 0$.

Now consider the general case, with $\partial N \neq \phi$. This will be reduced to the special case by showing that there is a compact oriented PL Z/r manifold M with $\partial M = \partial N$, and such that the action of Z/p on ∂N extends to a free action on M. The special case applies to $M \cup_\partial N$.

Since Z/p acts freely on ∂N, $\partial N/\psi$ defines a class in $\Omega_*^{PL}(K(Z/p, 1); Z/r)$. ∂N

bounds a Z/r manifold with a free Z/p action if this class is zero. The image of the class under the transfer homomorphism to $\Omega_*^{PL}(\text{point}; Z/r)$ is zero, since it is represented by ∂N which bounds N. If p and r are relatively prime, then the transfer is an isomorphism and ∂N bounds freely. This is sufficient for the application, since there r is a power of 2. In general the kernel of the transfer will be p-torsion, so an odd multiple of ∂N bounds. This shows that $w_p(kL) = 0$ for k odd, and therefore $w_p(L) = 0$.

§3. RELATION TO OTHER CLASSES

Let $W_T(Q)$ denote the torsion subgroup of the Witt group of the rationals. In[6, 7, 9] a characteristic class

$$\gamma_T(K) \epsilon H_{4*+k}(K, \partial K; W_T(Q))$$

is defined for an integrally oriented PL rational homology manifold. In[9] this is denoted $\lambda(K)$ (see[9], p. 54).

If K is a Z/p (hence $Z_{(p)}$) homology manifold $\gamma_T(K)$ lies in $H_{4*+k}(K, \partial K; W_T(Z_{(p)}))$. The reduction homomorphism $Z_{(p)} \to Z/p$ induces a homomorphism $W_T(Z_{(p)}) \to W(Z/p)$ whose image is the kernel of the rank homomorphism; the $Z/2$ used in 1.1. Let $\alpha : W_T(Z_{(p)}) \to Z/2$ denote the homorphism to the image.

PROPOSITION 3.1. *$h^p(K)$ is the Bockstein of $\hat{\alpha}(\gamma_T(K))$.*

Here the Bockstein is for the sequence of coefficient groups $0 \to Z \to Z \to Z/2 \to 0$ (or $0 \to Z_{(2)} \to Z_{(2)} \to Z/2 \to 0$), and $\hat{\alpha}$ is the change of coefficients homomorphism induced by α.

The proof is a simple comparison of the definitions, so is left to the reader.

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Department of Mathematics
State University of New York at Stony Brook
NY 11794, U.S.A.